

Noisy Processing and Distillation of Private Quantum States

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We provide a simple security proof for prepare and measure quantum key distribution protocols employing noisy processing and one-way postprocessing of the key. This is achieved by showing that the security of such a protocol is equivalent to that of an associated key distribution protocol in which, instead of the usual maximally entangled states, a more general *private state* is distilled. In addition to a more general target state, the usual entanglement distillation tools are employed (in particular, Calderbank-Shor-Steane-like codes), with the crucial difference that noisy processing allows some phase errors to be left uncorrected without compromising the privacy of the key.

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Entanglement has been the cornerstone of many quantum key distribution (QKD) security proofs to date: A prepare and measure protocol by which Alice and Bob generate a secret key can be shown to be secure exactly when an associated entanglement distillation protocol succeeds in producing a high-fidelity maximally entangled state. Secrecy of the key then follows since maximal entanglement can be shared only between two parties [1–8]. The resulting proofs are intuitive and allow QKD designers to incorporate current methods of quantum error correction and entanglement distillation.

Renner, Gisin, and Kraus adopt a more information-theoretic approach to QKD security with the surprising result that secure key can be established at noise levels beyond what seems possible in the entanglement-based picture [9]. By including a step in which Alice adds noise to her sifted key, the overall key rate can actually increase. The additional noise damages the correlations held by Alice and Bob but the key observation is that this noise may damage Eve's correlations even more. While the best known upper bounds for one-way distillable entanglement do not rule out the possibility of distilling EPR pairs for these noise levels, it is puzzling that this processing can generate key at rates well in excess of the best known entanglement distillation rates [10]. Thus, it has been unclear whether an entanglement-based security proof is possible for these protocols.

We find a resolution in the observation of [11] that maximally entangled states are not strictly necessary for generating secret keys. Instead, states leading to secret keys belong to the class of *private states*. These are composed of completely correlated systems A and B containing a uniformly distributed key, along with *shield* systems A' and B' . More precisely, γ is called a d -dimensional private state (or *pdit*) if there are unitaries $U^{(i)}$ and a *twisting operator* of the form $U_{\text{twist}} = \sum_j |jj\rangle\langle jj|_{AB} \otimes U_{A'B'}^{(j)}$, such that $\gamma = U_{\text{twist}}(|\Phi_d\rangle\langle\Phi_d|_{AB} \otimes \rho_{A'B'})U_{\text{twist}}^\dagger$ for some $\rho_{A'B'}$, where $|\Phi_d\rangle = \sum_{i=1}^d |ii\rangle/\sqrt{d}$. The twisting operator ensures

that, while Alice and Bob may not share a maximally entangled state, Eve's reduced state is independent of the key. This definition recalls an earlier result [12] that the secrecy of key created from entangled systems is not diminished by phase noise in the devices performing the entanglement distillation.

In [13] it was shown that a large number of low fidelity copies of a private state can sometimes be distilled to a high-fidelity private state with the same shield but smaller key system. However, it is not clear what class of QKD protocols can be coherently recast in the form considered by [13]. As we will see, the coherent version of the protocol of [9] is quite different from those of [13]—the initial adversarially distributed state will be noisy EPR pairs (with no shield), and the shield of the final private state arises due to Alice and Bob's noisy processing (Fig. 1).

In the following, we show that a prepare and measure QKD scheme with noisy processing and one-way postpro-

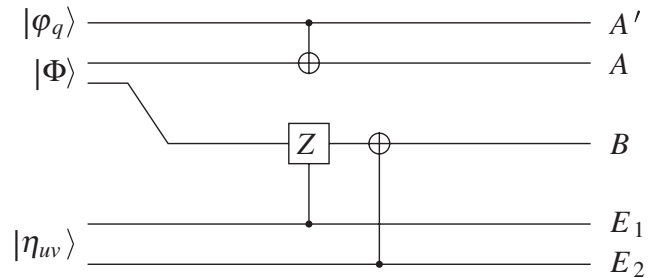


FIG. 1. The effective state held by Alice, Bob, and Eve after noisy processing, where $|\varphi_q\rangle = \sqrt{1-q}|0\rangle + \sqrt{q}|1\rangle$, $|\eta_{uv}\rangle = \sum_{u,v} \sqrt{p_{uv}} |uv\rangle$, and A' is the purification of the noise Alice adds. CSS-like error correction on the AB system is equivalent to classical error correction and privacy amplification on the key in the prepare and measure protocol, and securely provides key exactly when it maps many copies of the above state to a high-fidelity private state for all p_{uv} consistent with the estimated parameters. The shield consists of the A' systems together with the CSS code's syndrome bits held by Bob.

cessing is secure exactly when an associated pdit distillation protocol has high fidelity. This requires only minor modifications of the standard entanglement distillation argument. Indeed, in the coherent description of the noisy processing protocol the auxiliary system purifying the noise introduced by Alice will function as a shield, and the sifted key will become noisy EPR pairs in the key system of a noisy pdit. The error correction and privacy amplification required in the classical processing maps to a Calderbank-Shor-Steane (CSS)-like quantum code on the key system in the coherent protocol in the same way as found in [2]. If the CSS code performs a suitable amount of bit and phase error correction on the key system, Alice and Bob will be left with a high-fidelity private state. The crucial difference from previous entanglement-based security proofs is that Alice and Bob need not correct every phase error to guarantee security, and this savings will often more than compensate for the associated increase in the number of bit errors they must correct. In fact, we can establish key at bit-error rates up to 12.4% for the Bennett-Brassard-84 (BB84) protocol [14] and 14.1% for the six-state protocol, matching the rates of [9] and surpassing all previous thresholds from entanglement-based proofs.

Private state distillation.—We begin with a coherent reformulation of the BB84 and six-state protocols [1,2]; other protocols can be handled in a similar manner [8]. In both cases, Alice first prepares the state $|\Phi\rangle_{AB}$ and sends the B system to Bob. In BB84 (six state), each party then randomly measures in the X or Z basis (X , Y , or Z) and by public discussion they sift out those outcomes corresponding to the same basis choice. This is equivalent to Alice (Bob) sending a random bit in (measuring in) one of the bases at random, since the statistics of measurements as well as an eavesdropper Eve's dependence on their outcomes are identical in both cases. Alice and Bob then publicly compare a small fraction of the sifted key to estimate the noise level of the channel.

If the noise level is zero, the resulting length- n sifted key can be described coherently as $|\Phi\rangle_{AB}^{\otimes n}$. Otherwise, the most general noisy channels we need to consider are Pauli channels, since all subsequent operations will commute with a (hypothetical) measurement in the Bell basis which digitizes the actual noise into this form [1,6]. The only difference here to the original classicization argument of Lo and Chau [1] is that Alice flips some key bits, which also commutes with the Bell-state measurement. Attributing the noise to Eve, the key state is

$$\sum_{\mathbf{u}, \mathbf{v}} \sqrt{p_{\mathbf{uv}}} (\mathbb{I}_A \otimes X_B^{\mathbf{u}} Z_B^{\mathbf{v}}) |\Phi\rangle_{AB}^{\otimes n} |\mathbf{u}\rangle_{E_1} |\mathbf{v}\rangle_{E_2}, \quad (1)$$

where $p_{\mathbf{uv}}$ is the probability of error pattern $X^{\mathbf{u}} Z^{\mathbf{v}}$ described by length- n bit strings \mathbf{u} and \mathbf{v} . Furthermore, if Alice and Bob randomly permute their n systems, it is sufficient to consider noise that is independent and identically distributed (IID) for each transmitted qubit, given by rate p_{uv} . This follows from a slight variant of Lemma 3 of

[6] (see also [2,4]), the particulars of which we take up after the detailed analysis of the next section.

Alice and Bob now distill the key by performing bit error correction and privacy amplification (phase error correction). Before this, Alice adds IID noise to A , randomly applying X at rate q . This is described coherently as using an auxiliary system A' in the state $|\varphi\rangle_{A'} = \sqrt{1-q}|0\rangle + \sqrt{q}|1\rangle$ as the control system in a CNOT gate, yielding the state

$$\sum_{\mathbf{u}, \mathbf{v}, \mathbf{f}} \sqrt{p_{\mathbf{uv}} q_{\mathbf{f}}} |\mathbf{f}\rangle_{A'} (X_A^{\mathbf{f}} \otimes X_B^{\mathbf{u}} Z_B^{\mathbf{v}}) |\Phi\rangle_{AB}^{\otimes n} |\mathbf{u}\rangle_{E_1} |\mathbf{v}\rangle_{E_2}, \quad (2)$$

where $q_{\mathbf{f}} = q^{|\mathbf{f}|} (1-q)^{n-|\mathbf{f}|}$ for length- n bit string \mathbf{f} and $|\mathbf{f}|$ its Hamming weight. We can also think of Alice's error operator acting on Bob's system, since $X \otimes XZ$ and $\mathbb{I} \otimes XZX$ have the same effect on $|\Phi\rangle$.

Now Alice and Bob perform bit error correction using a linear error-correcting code. This step is the same as the usual analysis, since all bit errors must be corrected in the final key. The bit-error rate is $\tilde{p} = p_x(1-q) + q(1-p_x)$ for $p_x = \sum_v p_{1,v}$, so Alice and Bob must measure $nH_2(\tilde{p})$ parity syndromes, where H_2 is the binary Shannon entropy, in order to identify the error pattern with high probability. To simplify the resulting expressions, we use the method of decoupling error correction and privacy amplification [15], itself based on the breeding entanglement distillation protocol [16], whereby syndromes are collected in auxiliary entangled pairs.

Alice collects the bit parities in her halves of the ancilla states, measures them, and sends the result to Bob. Bob then coherently corrects system B and records the error in an ancilla system B' , producing

$$\sum_{\mathbf{u}, \mathbf{v}, \mathbf{f}} \sqrt{p_{\mathbf{uv}} q_{\mathbf{f}}} Z_{A'}^{\mathbf{v}} |\mathbf{f}\rangle_{A'} |\mathbf{u} + \mathbf{f}\rangle_{B'} Z_B^{\mathbf{v}} |\Phi\rangle_{AB}^{\otimes n} |\mathbf{u}\rangle_{E_1} |\mathbf{v}\rangle_{E_2}, \quad (3)$$

where $Z_{A'}^{\mathbf{v}}$ comes from interchanging $X_B^{\mathbf{f}}$ and $Z_B^{\mathbf{v}}$.

In the classical description of the protocol, this step requires Alice to encrypt her measurement outcomes with a one-time pad, preventing information leakage to Eve. This encryption requires a key, which in the coherent description is a private state, meaning Alice and Bob generally collect the parity syndromes in the key subsystems of private states, not in maximally entangled states as we have used. However, there is no loss of generality in using maximally entangled states in the formalism, since using private states raises no additional complications [13,17].

At this stage, the normal entanglement-based proof would proceed to correct all phase errors. This would not give the key rates of [9] as the extra noise would just reduce the rates from those of [2]. Instead, we come to the main observation of this Letter: not all phase errors must be corrected. After correcting enough, the resulting state will be close to a private state.

Examining Alice and Bob's state makes clear how this comes about. Tracing out Eve's systems, they hold

$$\rho = C_{A'B'} \left(\sum_{\mathbf{u}, \mathbf{v}} p_{\mathbf{u}\mathbf{v}} [\mathbf{u}]_{B'} [\varphi^{\mathbf{v}}]_{A'} Z_B^{\mathbf{v}} [\Phi]_{AB}^{\otimes n} Z_B^{\mathbf{v}} \right) C_{A'B'}^{\dagger}, \quad (4)$$

where $[\theta] = |\theta\rangle\langle\theta|$, $[\varphi^{\mathbf{v}}] = Z^{\mathbf{v}}|\varphi\rangle\langle\varphi|^{\otimes n}$, and we have used a CNOT $C_{A'B'}$ to write $[\mathbf{f}]_{A'}[\mathbf{u}] + [\mathbf{f}]_{B'}$ as $C_{A'B'}[\mathbf{f}]_{A'}[\mathbf{u}]_{B'}$.

By performing phase error correction at a reduced rate, the pattern of phase errors will not be uniquely identified, but only narrowed to a set \mathcal{V}_s indexed by the syndrome \mathbf{s} : $\mathcal{V}_s = \{\mathbf{v} | \text{syndrome}(\mathbf{v}) = \mathbf{s}\}$. The key point is that if the vectors $[\varphi^{\mathbf{v}}]$ for $\mathbf{v} \in \mathcal{V}_s$ were mutually orthogonal, we could define the unitary $D_{A'B} = \sum_{\mathbf{v} \in \mathcal{V}_s} [\varphi^{\mathbf{v}}]_{A'} \otimes Z_B^{\mathbf{v}}$ and use $U_{BA'B'} = D_{A'B} C_{A'B'}$ to untwist:

$$\begin{aligned} \rho' &= U_{BA'B'} \rho U_{BA'B'}^{\dagger} \\ &= [\Phi]_{AB}^{\otimes n} \otimes \left(\sum_{\mathbf{u}} p_{\mathbf{u}} [\mathbf{u}]_{B'} \sum_{\mathbf{v} \in \mathcal{V}_s} p_{\mathbf{v}|\mathbf{u}} [\varphi^{\mathbf{v}}]_{A'} \right). \end{aligned} \quad (5)$$

Since D is a controlled- Z gate, either system can be thought of as the control, so $D_{A'B} = \sum_{\mathbf{j}} U_{A'}^{(\mathbf{j})} \otimes [\mathbf{j}]_B$ for some unitaries $U^{(\mathbf{j})}$. $U_{BA'B'}$ is a twisting operation, so that Alice and Bob would share a private state. Keys derived from this state would be secret.

Detailed analysis.—To establish the secrecy of keys generated from ρ , recall the universally composable definition of security from [18,19]. A key K is called ϵ secure if the state ρ_{KE} of the key and eavesdropper satisfies $\|\rho_{KE} - \kappa \otimes \rho_E\|_1 \leq 2\epsilon$, where κ is a uniform mixture of all key values, shared by Alice and Bob. The latter state is a perfect key and this definition ensures that ρ_{KE} can safely be used for any further cryptographic purpose.

In the present context, the key is created by measuring systems A and B of ρ in the Z basis. As the untwisting operation is unitary and commutes with the measurement, whether it is performed before the measurement or after does not affect the key's security. When performing the untwisting operation on the unmeasured state results in a maximally entangled state on AB , the key generated will be perfectly secure. Similarly, if there is an untwisting operation mapping AB to within 2ϵ of a maximally entangled state, the key is ϵ secure [11].

For simplicity we consider independent amplitude and phase errors, with the case of correlated \mathbf{u} and \mathbf{v} following along similar lines. To construct an untwisting operation, it suffices to find a rank-one positive operator valued measure with elements $E_{\mathbf{v}}$ that can distinguish the $[\varphi^{\mathbf{v}}]$ with average error P_e no larger than $\epsilon^2/2$: $P_e = \langle P_e^{\mathbf{v}} \rangle = \sum_{\mathbf{v}, \mathbf{v}' \neq \mathbf{v}} p_{\mathbf{v}'} \langle \varphi^{\mathbf{v}} | E_{\mathbf{v}'} | \varphi^{\mathbf{v}} \rangle \leq \epsilon^2/2$, where $P_e^{\mathbf{v}}$ is the probability of decoding input state $[\varphi^{\mathbf{v}}]$ incorrectly. This problem was considered by [20] in the context of transmitting classical information over a quantum channel. Letting $\sigma = (1 - p_z)|\varphi\rangle\langle\varphi| + p_z Z|\varphi\rangle\langle\varphi|Z$, $p_z = \sum_{\mathbf{u}} p_{\mathbf{u}1}$, and $S(\sigma)$ be the entropy of σ , their results imply that with probability

$1 - \epsilon^2/2$ the elements of a randomly chosen subset $\mathcal{V}_s \subset \mathcal{V}$ of size $2^{n[S(\sigma) - \delta]}$ can be distinguished by the pretty-good measurement (PGM) with average error probability $\epsilon^2/2$, where ϵ decreases exponentially with n for arbitrarily small positive δ .

The PGM has rank-one elements by construction [21], so we have $E_{\mathbf{v}} = |\tilde{\theta}^{\mathbf{v}}\rangle\langle\tilde{\theta}^{\mathbf{v}}|$ for unnormalized $|\tilde{\theta}^{\mathbf{v}}\rangle$. Then we can append another auxiliary system A'' and consider the Neumark extension consisting of orthonormal states $|\theta^{\mathbf{v}}\rangle_{A'A''}$ in the joint Hilbert space $A'A''$ such that $A'A''\langle\theta^{\mathbf{v}}|\varphi^{\mathbf{v}}\rangle_{A'}|0\rangle_{A''} = A'\langle\tilde{\theta}^{\mathbf{v}}|\varphi^{\mathbf{v}}\rangle_{A'}$ [22]. With this, we can finally construct the untwisting operator $U = (\sum_{\mathbf{v}} [\theta^{\mathbf{v}}]_{A'A''} \otimes Z_B^{\mathbf{v}}) C_{A'B'}^{\dagger}$.

Letting $\tilde{\rho} = |0\rangle\langle 0| \otimes \rho$, the fidelity of $U\tilde{\rho}U^{\dagger}$ with $\rho' = [\Phi]_{AB}^{\otimes n} \otimes \sum_{\mathbf{u}, \mathbf{v}} p_{\mathbf{u}\mathbf{v}} [\theta^{\mathbf{v}}]_{A'A''} \otimes [\mathbf{u}]_{B'}$ is given by $F(U\tilde{\rho}U^{\dagger}, \rho') = \sum_{\mathbf{u}, \mathbf{v}} p_{\mathbf{u}\mathbf{v}} |\langle \varphi^{\mathbf{v}} | \tilde{\theta}^{\mathbf{v}} \rangle| = \langle \sqrt{P_s^{\mathbf{v}}} \rangle$, where $P_s^{\mathbf{v}}$ is the conditional probability of successful transmission of \mathbf{v} . Since $\langle \sqrt{P_s^{\mathbf{v}}} \rangle \geq \langle P_s^{\mathbf{v}} \rangle = 1 - P_e \geq 1 - \epsilon^2/2$, using the relation between trace norm and fidelity [23], we find $\|U\tilde{\rho}U^{\dagger} - \rho'\|_1 \leq 2\sqrt{1 - F^2} \leq 2\epsilon$, proving ϵ security.

A subtlety arises in the use of the Neumark extension in that our untwisting operation consists of controlled isometries rather than unitaries. However, the privacy of the key is uncompromised: while Eve may have knowledge of the shield system, as long as Alice and Bob hold the key and shield, the fact that they could be untwisted implies that Eve is ignorant of the key.

Above, we took \mathbf{u} and \mathbf{v} to be independent. If they are not, randomly choosing sets \mathcal{V}_s of size $2^{n[S(\sigma|u) - \delta]}$, where $S(\sigma|u)$ is the conditional entropy of σ given u , leads to an exponentially small average probability of decoding error for the PGM, and the rest of the argument remains unchanged [3]. Putting this all together, by using a random code Alice and Bob can select a subset \mathcal{V}_s of size $\approx 2^{nS(\sigma|u)}$. With probability exponentially close to 1, the untwisting operation can be constructed from the pretty-good measurement, ensuring the key is ϵ secure.

Finally, we must consider the effects of non-IID noise, e.g., arising from a coherent eavesdropping attack. By random sampling, Alice and Bob obtain an estimate $f_{u,v}^{\text{est}}$ of the fraction, or *type*, of Pauli errors $X^u Z^v$. Since the raw key bits are permutation invariant, $|f_{u,v}^{\text{est}} - f_{u,v}^{\text{true}}| \leq \epsilon$ with probability exponentially close (in n) to unity [24]. This allows us to prove that the above procedure is secure for *any* input state yielding estimate $f_{u,v}^{\text{est}}$, not just those subjected to IID noise. First, decompose the squared fidelity for an IID input state with error rate $p_{u,v} = f_{u,v}^{\text{est}}$ into a sum over possible types f : $F^2 = \sum_f \text{prob}(f|p = f^{\text{est}}) F_f^2$, where $\text{prob}(f|p = f^{\text{est}})$ is the probability of type f in the IID distribution, F_f is the fidelity our protocol produces on a uniform distribution over errors of type f , and we have suppressed the u, v indices. Since there are only polynomially many types, all those with non-negligible probability must have polynomially large probability and thus corresponding fidelities F_f which are exponentially close

to 1. Since types within ε of the rate p are among the probable types [25], this guarantees that the above procedure produces high-fidelity entangled output states (or securely aborts) for any input state yielding f^{est} .

Achievable key rates.—What key generation rates can be achieved by the protocols considered above? The bit-error-correction step consumes $nH_2(\tilde{p})$ previously established secret key bits, but in so doing produces n bit-error-free bits. The phase error correction must reduce the number of phase errors from $2^{nH(v|u)}$ to $2^{nS(\sigma|u)}$ (which can be accomplished by a random phase code with $n[H(v|u) - S(\sigma|u)]$ syndrome bits) in order to ensure that Alice and Bob could untwist the state, so we find an overall rate of $1 - H_2(\tilde{p}) - [H(v|u) - S(\sigma|u)]$, or

$$R = 1 - H_2(\tilde{p}) - \sum_u p_u [H_2(p_{1|u}) - H_2(\lambda_u^+)], \quad (6)$$

where $\lambda_u^+ = \frac{1}{2}(1 + \sqrt{1 - 16q(1-q)p_{1|u}(1-p_{1|u})})$ is the larger eigenvalue of $\sigma_u = (1 - p_{1|u})|\varphi\rangle\langle\varphi| + p_{1|u}|Z\rangle\langle Z|$.

In the BB84 protocol, bit and phase errors are equal but uncorrelated, meaning $p_{1|u} = p_z = p_x = p_{1|v}$, from which we find an error threshold of 12.4% by letting $q \rightarrow 1/2$. In the six-state protocol all Pauli errors occur at the same rate, giving a threshold error rate of 14.1%.

Discussion.—We have shown that one-way key distribution protocols employing noisy processing can be seen as distillation protocols for private states where the purification of the added noise functions as part of the shield and the error-correction and privacy amplification steps map to a CSS code in the usual way. This extends the entanglement distillation paradigm initiated in [1,2], providing a cleaner and less technical security proof for the protocols of [9]. Further, by formulating the protocol in this way, we gain insight into the mechanism by which addition of noise improves key rates, namely, by deflecting Eve's correlations with Alice and Bob to the shield and away from the key.

In the security proof of the six-state protocol [3], building on the work of [26], Lo showed that a degenerate error-correcting code could be used to improve the threshold error rate from 12.6% to 12.7%. Further progress in this direction can be found in [27], where we report on the combination of that method with the noisy processing studied here, showing that the threshold error rate of BB84 can be increased from 12.4% to 12.9%. We believe our findings will point towards new methods of key distillation and analogous methods of private state distillation, furthering the fruitful exchange between privacy amplification and entanglement distillation.

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